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Approximating with Lipschitz Controls

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For nonlinear nonconvex control systems in finite dimensional state spaces we approximate measurable controls by Lipschitz controls. We give explicit approximation rates. It turns out that the corresponding trajectories converge uniformly on bounded time intervals and that the approximation is of order $O(M^{-1/2})$, as $M \to \infty$, where $M > 0$ is the Lipschitz constant of the Lipschitz controls.

Key Words: nonlinear control system; Lipschitz controls.

1. INTRODUCTION

We consider ordinary differential equations with time-varying vector fields. At any time we can choose a particular vector field within a prescribed class.

If the time-dependence of the vector fields is *measurable*, the set of possible trajectories is richer than in the case that only *Lipschitz continuous* changes of the vector fields are allowed. However, as the Lipschitz constant is growing, the latter set of trajectories approximates the former one. In this paper we investigate the order of approximation that can be expected. This is done in the context of nonlinear nonconvex control systems

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x, \qquad u(t) \in \Omega \tag{1}$$

in the Euclidean space $\mathbb{R}^n$. The time-varying parameter $t \mapsto u(t)$ determines the time-dependence of the vector fields. We call it a *control*.

As usual, a solution $t \mapsto x(t) = x(t; x, u(\cdot))$ is an absolutely continuous function with $x(0) = x$ such that the differential equation of (1) is satisfied for (Lebesgue) almost all times $t \in [0, \infty)$.

We define two families of controls:

$$\mathcal{U} := \{u(\cdot) \colon [0, \infty) \to \Omega : u(\cdot) \text{ (Lebesgue) measurable}\}$$

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and, for a Lipschitz constant $M>0$,

$$\mathcal{U}_M := \{u(\cdot)\colon [0,\infty) \to \Omega : \|u(t)-u(s)\| \leqslant M\,\|t-s\| \text{ for all } t, s \in [0,\infty)\}.$$

It is well known that, in a weak sense, $\bigcup_{M>0} \mathcal{U}_M$ is dense in $\mathcal{U}$ and that any trajectory produced by a measurable control $u(\cdot) \in \mathcal{U}$ can be approximated by trajectories produced by Lipschitz controls $u_M(\cdot) \in \bigcup_{M>0} \mathcal{U}_M$, provided that the control range Ω is connected.

We are interested in explicit approximation rates and show that, for given initial value $x \in \mathbb{R}^n$ and time interval $[0, H]$, the estimate

$$\sup_{u(\cdot)\in\mathcal{U}} \Big(\inf_{w(\cdot)\in\mathcal{U}_M} \Big(\max_{t\in[0,H]} \|x(t,x,u(\cdot)) - x(t,x,w(\cdot))\| \Big)\Big) = O(M^{-1/2}),$$

as $M \to \infty$, is valid, see Theorem 2.4. For a certain class of nonlinear convex systems the order can be improved to $O(M^{-1})$, as $M \to \infty$, see Theorem 4.4.

The approximation of measurable controls by more regular controls is not only of theoretical but also of practical interest. Basically two classes of regular controls usually are considered. Firstly, the class of piecewise continuous controls and within this group the piecewise constant controls. Secondly, the class of Lipschitz controls. Whereas the approximation order for the former class has been investigated in a variety of articles, see [4, 5, 6, 8, 9], the latter class seems to lack a comparable investigation.

The question of approximating with Lipschitz controls is of some importance for practical problems, since for most mechanical systems measurable controls, or even piecewise constant controls, cannot be realized, due to the inertness of the controlling mechanism, see [3].

Furthermore, Lipschitz controls play a prominent part in game theory, where they are used to characterize the existence of a value. However, in this context one usually works with compactness arguments in order to achieve a convergence of the trajectories, see [2].

The paper is organized as follows. In the second section we give the setting and state and prove the main result, Theorem 2.4. The third section consists of an example showing the optimality of the achieved approximation order. Finally, in the last section we give some additional conditions, under which an improved approximation order can be achieved.

2. THE NONLINEAR NONCONVEX CASE

The setting is as follows.

- The control range Ω is a compact metric space.

- The vector fields $x \mapsto f(x, \omega)$ are uniformly Lipschitz continuous on $\mathbb{R}^n$ with Lipschitz constant $L \geqslant 0$.
- The map $(x, \omega) \mapsto f(x, \omega)$ is continuous on $\mathbb{R}^n \times \Omega$.
- For any time $H > 0$ and any initial value $x \in \mathbb{R}^n$ there is a constant $P \geqslant 0$ such that $\|f(y, \omega)\| \leqslant P$ for all $y \in Y := \{x(t; x, u(\cdot)) : t \in [0, H], u(\cdot) \in \mathcal{U}\}$ and all $\omega \in \Omega$.

These requirements ensure the unique existence of a trajectory for any initial value $x \in \mathbb{R}^n$ and any measurable control function $u(\cdot) \in \mathcal{U}$, see [7]. The last point is redundand, since for any initial state $x \in \mathbb{R}^n$ and any interval $[0, H]$ the set of reachable states Y is bounded in $\mathbb{R}^n$ (by the continuity properties of the vector fields together with the compactness of the control range). Nevertheless, we find it more convenient to prescribe such a bound $P \geqslant 0$ explicitly.

For $\omega \in \Omega$ we set

$$\mathcal{U}_M^\omega := \{u(\cdot) \in \mathcal{U}_M : \ u(0) = \omega\}.$$

The following assumption is essential.

Assumption 2.1.

- There is a time $T \geqslant 0$ such that for all $\omega_1, \omega_2 \in \Omega$ the inclusion

$$\mathcal{U}_1^{\omega_2} \subset \{\omega(\cdot) \in \mathcal{U}_1 : w(\cdot) = u(T + \cdot) \text{ for some } u(\cdot) \in \mathcal{U}_1^{\omega_1}\}$$

is valid.

Obviously, this implies connectedness of the control range Ω, in the sense that any two points $\omega_1, \omega_2 \in \Omega$ can be connected by a continuous path within Ω. Accordingly, we obtain for all $M > 0$:

$$\mathcal{U}_M^{\omega_2} \subset \left\{w(\cdot) \in \mathcal{U}_M : w(\cdot) = u\left(\frac{T}{M} + \cdot\right) \text{ for some } u(\cdot) \in \mathcal{U}_M^{\omega_1}\right\}.$$

For the right-hand side we shortly write

$$F(y) := \{f(y, \omega) : \ \omega \in \Omega\}$$

for $y \in Y$ and consider for $S > 0$ and $\omega \in \Omega$ the set-valued averages

$$F_M(S, y, \omega) = \left\{\frac{1}{S} \int_0^S f(y, u(t))\, dt : \ u(\cdot) \in \mathcal{U}_M^\omega\right\}.$$

By Caratheodory's theorem the set-valued averages converge, in the Hausdorff sense, to the closed convex hull of the right-hand side

$$\operatorname{conv} F(y) = \lim_{S\to\infty} F_M(S, y, \omega).$$

In order to obtain an approximation rate, we make use of a generalized version of Caratheodory's theorem.

LEMMA 2.2. *For all* $v \in \operatorname{conv} F(y)$ *and* $v_0 \in F(y)$ *there are* $v_1, ..., v_n \in F(x)$ *and* $\lambda_0, ..., \lambda_n \in [0, 1]$ *with* $\sum_{i=0}^n \lambda_i = 1$, *such that*

$$v = \sum_{i=0}^{n} \lambda_i v_i.$$

Proof. This follows from a more general statement in [1]. ∎

We recall the definition of the Hausdorff metric for compact subsets $A, B \subset \mathbb{R}^n$. We set $A^e := \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) \leqslant e\}$. Then the Hausdorff metric can be written as

$$\mathrm{d_H}(A, B) := \inf\{e > 0 : A \subset B^e \text{ and } B \subset A^e\}.$$

PROPOSITION 2.3. *Let* $M > 0$ *and* $y \in Y$. *For all* $\omega \in \Omega$ *and all* $S > 0$ *we can estimate*

$$\mathrm{d_H}(\operatorname{conv} F(y), F_M(S, y, \omega)) \leqslant \frac{2PTn}{SM}.$$

Proof. Denoting by $B_a(0)$ the closed ball with center $0 \in \mathbb{R}^n$ and radius $a \geqslant 0$ we obviously have for all $\omega \in \Omega$:

$$F_M(S, y, \omega) \subset \operatorname{conv} F(y) + B_{(2PTn)/(SM)}(0).$$

For the converse estimate, let $v \in \operatorname{conv} F(y)$. By Lemma 2.2, for $\omega_0 := \omega$, there are $\omega_1, ..., \omega_n \in \Omega$ and $\lambda_0, ..., \lambda_n \in [0, 1]$ with $\sum_{i=0}^n \lambda_i = 1$, such that

$$v = \sum_{i=0}^{n} \lambda_i f(x, \omega_i).$$

For $S \in (0, (nT)/M]$ the required estimate obviously is valid. For $S > (nT)/M$ we divide the interval $[0, S]$ into $2n+1$ subintervals $[0, \mu_0]$, $[\mu_0, T/M + \mu_0]$, $[T/M + \mu_0, T/M + \mu_0 + \mu_1]$, ..., $[nT/M + \mu_0 + \cdots + \mu_{n-1}, nT/M + \mu_0 + \cdots + \mu_n]$ with

$$\mu_i := \lambda_i (S - nT/M).$$

According to Assumption 2.1 we choose a control $u(\cdot) \in \mathscr{U}_M^\omega$ such that

$$u(t) = \omega_i \qquad \text{for} \quad t \in \left[iT/M + \sum_{k=0}^{i-1} \mu_k, iT/M + \sum_{k=0}^{i} \mu_k \right].$$

Then we can write

$$\begin{aligned} v(S) &:= \frac{1}{S} \int_0^S f(y, u(t))\, dt \\ &= \frac{1}{S} \left((S - nT/M) \sum_{i=0}^{n} \lambda_i f(y, \omega_i) + (nT/M)\, v^* \right) \\ &= \frac{1}{S} ((S - nT/M)\, v + (nT/M)\, v^*), \end{aligned}$$

where $v^* \in \operatorname{conv} F(y)$. Thus we found an element $v(S) \in F_M(S, y, \omega)$ with

$$\| v(S) - v \| \leqslant \frac{2PnT}{SM}$$

and the proof is finished. ∎

THEOREM 2.4. *Let* $M > 0$ *and* $H \geqslant 0$. *For any measurable control* $u(\cdot) \in \mathscr{U}$ *and any* $\omega \in \Omega$ *there is a Lipschitz control* $w(\cdot) \in \mathscr{U}_M^\omega$ *such that*

$$\max_{t \in [0, H]} \| x(t, x, w(\cdot)) - x(t, x, u(\cdot)) \| \leqslant \frac{1}{\sqrt{M}} 4e^{LH} \sqrt{(PLH + P)\, HPTn}.$$

Proof. We write shortly $x(t) = x(t, x, u(\cdot))$. For an $S_M > 0$ (to be specified later) we divide the interval $[0, H]$ into subintervals $[t_k, t_{k+1}]$, for $k = 0, ..., [(HM)/S_M]$, of lenght $t_{k+1} - t_k = S_M/M$, that is $t_k = (kS_M)/M$. Then we have

$$x(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} f(x(t), u(t))\, dt.$$

We define a sequence in $\mathbb{R}^n$ by $x_0 = x$ and

$$x_{k+1} = x_k + \int_{t_k}^{t_{k+1}} f(x_k, u(t))\, dt.$$

Then we obtain

$$\|x_k - x(t_k)\| \leqslant \frac{S_M}{M} PLHe^{LH}.$$

To this end we observe that

$$\|x_{k+1} - x(t_{k+1})\| \leqslant \|x_k - x(t_k)\| + \frac{S_M}{M} L \left(\frac{S_M}{M} P + \|x_k - x(t_k)\| \right)$$

$$= \|x_k - x(t_k)\| \left(1 + \frac{LS_M}{M} \right) + \frac{S_M^2}{M^2} LP.$$

We conclude that

$$\|x_k - x(t_k)\| \leqslant \frac{S_M^2}{M^2} LP \sum_{i=0}^{k-1} \left(1 + \frac{LS_M}{M} \right)^i$$

$$\leqslant \frac{S_M^2}{M^2} LP \left\lceil \frac{HM}{S_M} \right\rceil \left(1 + \frac{LS_M}{M} \right)^{HM/S_M}$$

$$\leqslant \frac{S_M}{M} PLHe^{LH}.$$

Alternatively we can write with $v_k := \frac{M}{S_M} \int_{t_k}^{t_{k+1}} f(x_k, u(t))\, dt \in \operatorname{conv} F(x_k)$:

$$x_{k+1} = x_k + \frac{S_M}{M} v_k.$$

We define a new sequence in $\mathbb{R}^n$ by $y_0 = x$ and

$$y_{k+1} = y_k + \frac{S_M}{M} w_k,$$

where $w_k \in F_M(S_M/M, y_k, \omega_k)$ is chosen such that

$$\|v_k - w_k\| \leqslant L \|x_k - y_k\| + \frac{2PTn}{S_M}.$$

Then we can estimate

$$\|y_{k+1} - x_{k+1}\| \leqslant \left(1 + \frac{LS_M}{M}\right) \|y_k - x_k\| + \frac{S_M}{M} \frac{2PTn}{S_M}.$$

Hence

$$\|y_k - x_k\| \leqslant \frac{S_M}{M} \frac{2PTn}{S_M} \sum_{i=0}^{k-1} \left(1 + \frac{LS_M}{M}\right)^i \leqslant H \frac{2PTn}{S_M} e^{LH}.$$

On the other hand the $w_k \in F_M(S_M/M, y_k, \omega_k)$ are produced by Lipschitz controls $u_k(\cdot) \in \mathscr{U}_M$ with $u_0(0) = \omega_0 := \omega$ and $u_k(\tau_k) = \omega_k := u_{k-1}(\tau_k)$. These controls define a Lipschitz control $w(\cdot) \in \mathscr{U}_M^\omega$, at least for $t \in [0, H]$. Let $t \mapsto y(t) := x(t, x, w(\cdot))$ be the corresponding trajectory. Then we can estimate as above

$$\|y(t_k) - y_k\| \leqslant \frac{S_M}{M} PLHe^{LH}.$$

Considering that

$$\max\{\|x(t) - x(t_k)\|, \|y(t) - y(t_k)\|\} \leqslant \frac{S_M}{M} P,$$

for $t \in [t_k, t_{k+1}]$, we finally obtain

$$\begin{aligned}
\|y(t) - x(t)\| &\leqslant 2 \frac{S_M}{M} PLHe^{LH} + 2 \frac{S_M}{M} P + H \frac{2PTn}{S_M} e^{LH} \\
&\leqslant 2e^{LH} \left(\frac{S_M}{M} (PLH + P) + \frac{1}{S_M} HPTn\right) \\
&\leqslant 4e^{LH} \frac{1}{\sqrt{M}} \sqrt{(PLH + P)\, HPTn},
\end{aligned}$$

where the last inequality is obtained by setting

$$S_M := \frac{\sqrt{MHPT}}{\sqrt{PLH + P}}$$

and the proof is finished. ∎

3. AN EXAMPLE

We present an example that shows that the approximation order $O(M^{-1/2})$, as $M \to \infty$, stated in Theorem 2.4 is optimal. Consider the system in $\mathbb{R}^2$

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} f_1(u_1(t)) \\ f_2(u_2(t)) \end{pmatrix},$$

where the control range $\Omega \subset \mathbb{R}^2$ is a union $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ with

$$\Omega_1 := [-2, -1] \times [0, 1], \qquad \Omega_2 := [-1, 1] \times \{1\},$$
$$\Omega_3 := [1, 2] \times [0, 1].$$

Hence, for a fixed control value $(\omega_1, \omega_2) \in \Omega$, the corresponding vector field is constant. We define

$$f_1(\omega_1) := \begin{cases} \omega_1 + 1 & \text{for} \quad \omega_1 \in [-2, -1] \\ 0 & \text{for} \quad \omega_1 \in [-1, 1], \\ \omega_1 - 1 & \text{for} \quad \omega_1 \in [1, 2] \end{cases} \qquad f_2(\omega_2) := \omega_2.$$

For simplicity we choose the initial state $x = (x_1, x_2)^T = (0, 0)^T \in \mathbb{R}^2$ and the time horizon $[0, H] = [0, 1]$.

Remark 3.1. In the sequel we make use of a more restrictive version of the Landau symbol O. For $\lambda \in \mathbb{R}$ we say that a function $M \mapsto h(M) \in \mathbb{R}$ is of order $\tilde{O}(M^{\lambda})$, as $M \to \infty$, if there are two constants $0 < c_1 \leqslant c_2$ such that for $M > 0$ large enough the estimates $c_1 M \leqslant |h(M)|$ and $|h(M)| \leqslant c_2 M$ are valid. In contrast, the standard Landau symbol O requires the second estimate only.

Now we assume that there is a $\kappa > 1/2$ such that

$$\sup_{u(\cdot) \in \mathcal{U}} \Big(\inf_{w(\cdot) \in \mathcal{U}_M} \Big(\max_{t \in [0, H]} \|x(t, x, u(\cdot)) - x(t, x, w(\cdot))\| \Big) \Big) = O(M^{-\kappa}). \quad (2)$$

For $M > 0$ and $\varepsilon \in (0, \kappa)$ we define a periodic measurable control $t \mapsto u(t) = (u_1(t), u_2(t))^T$ (with period $4M^{-\kappa+\varepsilon}$) by

$$u_1(t) := \begin{cases} 2 & \text{for} \quad t \in [M^{-\kappa+\varepsilon}, 3M^{-\kappa+\varepsilon}) \\ -2 & \text{for} \quad t \in [3M^{-\kappa+\varepsilon}, 5M^{-\kappa+\varepsilon})' \end{cases} \qquad u_2(t) := 0 \quad \text{for} \quad t \in \mathbb{R}.$$

This control produces a periodic trajectory $t \mapsto x(t; x, u(\cdot))$ with $x_2(t; x, u(\cdot)) \equiv 0$ and $t \mapsto x_1(t; x, u(\cdot))$ periodically oscillating between the values

$-M^{-\kappa+\varepsilon}$ and $+M^{-\kappa+\varepsilon}$. Since $\varepsilon>0$, for any $c>0$, we have $2M^{-\kappa+\varepsilon}> cM^{-\kappa}$ for $M>0$ large enough. Hence, for $M>0$ large enough, the x_1-trajectory only can be approximated by Lipschitz controls $w(\cdot)=(w_1(\cdot), w_2(\cdot))\in\mathcal{U}_M$ if $t\mapsto w_1(t)$ changes the sign. The number of changes is of order $\tilde{O}(M^{\kappa-\varepsilon})$, as $M\to\infty$. Notice that any change of sign causes a vertical deviation of order $\tilde{O}(M^{-1})$, as $M\to\infty$, since the Lipschitz control has to pass through Ω_2. So, for any $\varepsilon\in(0,\kappa)$, we collect an overall vertical deviation of order $\tilde{O}(M^{-1+\kappa-\varepsilon})$, as $M\to\infty$, which is a contradiction to (2).

4. THE NONLINEAR CONVEX CASE

The previous example shows that, even for convex right-hand sides $F(y)=\{f(y,\omega):\ \omega\in\Omega\}$, the approximation order $O(M^{-1/2})$, as $M\to\infty$, is optimal. Indeed, the velocity set for this example can be easily calculated: $F(y)=[-1,1]\times[0,1]$.

In order to achieve better approximation rates we have to impose some additional conditions on the mechanism that prescribes how control values $\omega\in\Omega$ are related to vector fields $f(\cdot,\omega)$. Notice that the map

$$\alpha\colon \Omega\to C(Y,\mathbb{R}^n),\qquad \omega\mapsto f(\cdot,\omega)$$

is continuous, if we equip $C(Y,\mathbb{R}^n)$ with the uniform topology.

Assumption 4.1.

- The set of all admissible vector fields on Y, $\alpha(\Omega)$, is convex.
- There is a Lipschitz continuous map $\beta\colon \alpha(\Omega)\to\Omega$ such that $\beta\circ\alpha=\mathrm{id}_\Omega$. We denote by $C\geqslant 0$ its Lipschitz constant.

Assumption 4.1 is stronger than Assumption 2.1. To see this, we just have to use the convexity of the set of vector fields and the Lipschitz continuity of β. Then we can set $T:=2PC$ and obtain the condition in Assumption 2.1.

Remark 4.2. Assumption 4.1 is satisfied for control affine systems given by $f(x,(\omega_1,...,\omega_m)):=f_0(x)+\sum_{i=1}^m \omega_i g_i(x)$ if the control range $\Omega\subset\mathbb{R}^m$ is convex and if the vector fields $g_1,...,g_m$ are linearly independent.

For $K>0$ and $\omega\in\Omega$ we consider the backward averaged system

$$\dot{z}(t)=K\int_0^{1/K} f(z(t),u(t-s))\,ds,\qquad z(0)=x,\qquad u(t)\in\Omega,\tag{3}$$

where we set $u(t):=\omega$ for $t\in(-\infty,0)$.

LEMMA 4.3. *Let $t \mapsto x(t)$ be the trajectory of* (1) *and $t \mapsto z(t)$ the trajectory of* (3), *both obtained with the same control $u(\cdot) \in \mathcal{U}$ and with the same initial value $x \in \mathbb{R}^n$. Then for all $H \geqslant 0$ we can estimate*

$$\max_{t \in [0, H]} \|x(t) - z(t)\| \leqslant \frac{1}{K} (3PLH + 2P + HP)\, e^{LH}.$$

Proof. We divide the interval $[0, H]$ into subintervals $[t_k, t_{k+1}]$, for $k = 0, ..., [HK]$, of lenght $t_{k+1} - t_k = 1/K$, that is $t_k = k/K$. We additionally set $t_{-1} := -1/K$. We define a sequence in $\mathbb{R}^n$ by $x_0 = x$ and

$$x_{k+1} = x_k + \int_{t_k}^{t_{k+1}} f(x_k, u(t))\, dt.$$

Then we obtain as in the proof of Theorem 2.4

$$\|x(t_k) - x_k\| \leqslant \frac{1}{K} PLHe^{LH}.$$

The solution of (3) fulfils

$$z(t_{k+1}) = z(t_k) + \int_{t_k}^{t_{k+1}} K \int_0^{1/K} f(z(t), u(t-s))\, ds\, dt.$$

We define a sequence in $\mathbb{R}^n$ by $z_0 := x$ and

$$z_{k+1} := z_k + \int_{t_k}^{t_{k+1}} K \int_0^{1/K} f(z_k, u(t-s))\, ds\, dt.$$

In the same way as in the proof of Theorem 2.4 we obtain the estimate

$$\|z(t_k) - z_k\| \leqslant \frac{1}{K} PLHe^{LH}.$$

With the transformation $r := t - s$ and diagonal integration we can write

$$z_{k+1} = z_k + \int_{t_{k-1}}^{t_k} K(r - t_{k-1})\, f(z_k, u(r))\, dr + \int_{t_k}^{t_{k+1}} K(t_{k+1} - r)\, f(z_k, u(r))\, dr.$$

Now, we estimate the distance $\Delta_k := \|x_k - z_k\|$. To this end we write

$$z_{k+1} = z_0 + \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} f(z_i, u(r))\, dr$$
$$+ \int_{t_k}^{t_{k+1}} K(t_{k+1} + r)\, f(z_k, u(r))\, dr - \int_{t_{-1}}^{t_0} K(r - t_{-1})\, f(z_0, u(r))\, dr$$
$$+ \sum_{i=0}^{k} \left(\int_{t_i}^{t_{i+1}} K(r - t_i)\, f(z_{i+1}, u(r))\, dr - \int_{t_i}^{t_{i+1}} K(r - t_i)\, f(z_i, u(r))\, dr \right).$$

We obtain $\Delta_1 \leqslant \frac{2P}{K}$ and for all $k = 1, ..., [KH]$:

$$\Delta_{k+1} \leqslant \Delta_k + \frac{L}{K} \Delta_k + \frac{(L+1)\, P}{K^2} L$$
$$\leqslant \Delta_k \left(1 + \frac{L}{K}\right) + \frac{(L+1)\, P}{K^2}.$$

We conclude that

$$\Delta_{k+1} \leqslant \frac{2P}{K} \left(1 + \frac{L}{K}\right)^k + \frac{(L+1)\, P}{K^2} \sum_{i=0}^{k-1} \left(1 + \frac{L}{K}\right)^i.$$

Hence we obtain

$$\Delta_{k+1} \leqslant \frac{2P}{K} e^{LH} + \frac{(L+1)\, PH}{K} e^{LH}$$

and the proof is finished. ∎

THEOREM 4.4. *Let $M > 0$ and $H \geqslant 0$. For any measurable control $u(\cdot) \in \mathscr{U}$ and any $\omega \in \Omega$ there is a Lipschitz control $w(\cdot) \in \mathscr{U}_M^\omega$ such that*

$$\max_{t \in [0, H]} \|x(t, x, w(\cdot)) - x(t, x, u(\cdot))\| \leqslant \frac{1}{M} 2PC(3PLH + 2P + HP)\, e^{LH}.$$

Proof. According to Lemma 4.3 we have

$$\max_{t \in [0, H]} \|x(t) - z(t)\| \leqslant \frac{1}{K} (3PLH + 2P + HP)\, e^{LH}. \tag{4}$$

for solutions of the system (3). On the other hand the map

$$\gamma\colon [0, H] \to C(Y, \mathbb{R}^n), \qquad t \mapsto K \int_0^{1/K} f(\cdot, u(t-s))\, ds$$

obviously is Lipschitz continuous with Lipschitz constant $2KP \geqslant 0$. By Assumption 4.1 we even have $\gamma(t) \in \alpha(\Omega)$ and for all $t \in [0, H]$ there is a unique $\omega = (\beta \circ \gamma)(t) \in \Omega$ with

$$K \int_0^{1/K} f(\cdot, u(t-s))\, ds = f(\cdot, \omega).$$

The composition map

$$\beta \circ \gamma : [0, H] \to \Omega$$

is Lipschitz continuous with Lipschitz constant $2KPC \geqslant 0$. Furthermore we can define a Lipschitz control $w(t) := (\beta \circ \gamma)(t)$. Then $w(0) = \omega$ and $t \mapsto w(t)$ produces the trajectory of (3). Hence, for $PC > 0$, the claim follows by setting $K := M/(2PC)$ in (4). For $PC = 0$, the class of admissible vector fields is single-valued and nothing is to show. ∎

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